Uncertainty Analysis in Software Reliability Modeling by Bayesian Analysis and Maximum-Entropy Principle

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Outline

• Introduction
• Reliability Modeling & Uncertainty Problems
• Uncertainty Analysis by Maximal Entropy Principle and Bayesian Analysis
• Case Study
• Future Research
• Conclusion

Introduction

• Software Reliability Models
  – Parametric & Non-Parametric

• Uncertainty analysis
  – Uncertainty in Parameters (Random Variables)
  – Uncertainty in Model Outputs/Results
  – Variance, confidence interval, percentiles, bounds etc

• Bayesian Approach
  – Lack of failure data in software testing
  – Posterior=Priori * Likelihood of Observations

• Maximal Entropy Principle
  – Physical principle of Entropy
  – Priori (Expert’s Suggestions, Historical Data, Information, etc.)
Software Reliability Modeling

• MLE (Maximum Likelihood Estimate)
  – Test the component and record the failure times
    \[ t_k \quad (k = 1, 2, \ldots, n) \]
    \[ s_k = \sum_{i=1}^{k} t_i \]
  – Compute the joint density or likelihood function
    \[ f_S(s_1, s_2, \ldots, s_n) = \exp\{-m(s_n)\} \cdot \prod_{i=1}^{n} \lambda(s_i) \]
  – Get parameters that maximize likelihood function

• Reliability model with multiple components
  – Markov model, Stochastic Petri Net, fault tree analysis, reliability block diagram, etc.
  – Function of parameters of those components
Uncertainty Problems

• Randomness
  – Estimators and Observations
  – Test procedures and strategies
  – Model Selection

• Error of Parameters Estimations
  – Lack of Failure Data
  – Noise

• Accumulations of Uncertainty
  – Multiple components (Modular Software, or System)
  – Multiple procedures (Hierarchical Modeling)
Bayesian Approach

Standard BA Steps:

1) The parameters modeling a component are denoted by \( \bar{a} = \{a_1, a_2, \ldots, a_m \} \). The mean value function of the model is denoted by \( m(t \mid \bar{a}) \) and the failure intensity function by \( \lambda(t \mid \bar{a}) \).

2) The prior joint distribution of the parameters is denoted by \( p(\bar{a}) \) which is unknown.

3) The component is tested and a total of \( n \) failures have been observed. Let \( s_k \) denote the time to the \( k \)-th failure \( k = 1, 2, \ldots, n \), and \( \bar{s} = \{s_1, s_2, \ldots, s_n \} \) that are conditionally independent. Then, given the prior distribution and observations, the posterior distribution can be obtained by

\[
p(\bar{a} \mid \bar{s}) \propto p(\bar{a}) \cdot p(\bar{s} \mid \bar{a})
\]

where

\[
p(\bar{s} \mid \bar{a}) = \exp \{-m(s_n \mid \bar{a})\} \cdot \prod_{i=1}^{n} \lambda(s_i \mid \bar{a})
\]
Maximum-Entropy Principle

- **Entropy**: measures the disorder, and always tends to the maximum

- **Entropy Function**: 
  \[ H(f) \equiv -\int_{D_y} f(y) \cdot \ln[f(y)] dy \]
  where \( Y \) be a random variable with pdf \( f \), defined on \( D_y \subset \mathbb{R} \)

- **Constraints**:
  \[
  \int_{D_y} f(y) \cdot g_r(y) dy = \bar{g}_r \quad (r=1,2,\ldots,m) \quad \text{(Prior Information and Knowledge)}
  \]
  \[
  \int_{D_y} f(y) dy = 1
  \]

- **MEP**: Maximize \( H(f) \) subject to the constraints
Extract Information from MEP

- **Discrete**

  Provided information on the mean values $F_k$ of certain function $f_k(x)$

  $$\sum_{i=1}^{n} \Pr(x_i \mid I)f_k(x_i) = F_k \quad k = 1,\ldots,m$$

  From MEP, we can get the priori of parameters as

  $$\Pr(x_i \mid I) = \frac{1}{Z(\lambda_1,\ldots,\lambda_m)} \exp[\lambda_1 f_1(x_i) + \ldots + \lambda_m f_m(x_i)]$$

  where

  $$Z(\lambda_1,\ldots,\lambda_m) = \sum_{i=1}^{n} \exp[\lambda_1 f_1(x_i) + \ldots + \lambda_m f_m(x_i)]$$

- **Continuous**

  $$\int p(x) f_k(x) dx = F_k$$

  From MEP, we can get the priori of parameters as

  $$p(x) = \frac{1}{Z(\lambda_1,\lambda_2,\ldots,\lambda_m)} \exp[\lambda_1 f_1(x) + \ldots + \lambda_m f_m(x)]$$

  where

  $$Z(\lambda_1,\ldots,\lambda_m) = \int \exp[\lambda_1 f_1(x) + \ldots + \lambda_m f_m(x)] dx$$
Measures for Uncertainty

- Marginal density function
  \[ p_i(a_i \mid \bar{s}) = \int \int \cdots \int p(\bar{a} \mid \bar{s}) \cdot d(a_1) \cdots d(a_{i-1})d(a_{i+1}) \cdots d(a_m) \]

- Mean value of corresponding parameter
  \[ \hat{a}_i = E(a_i) = \int_{-\infty}^{+\infty} a_i \cdot p_i(a_i \mid \bar{s}) \cdot d(a_i) \]

- Variance of the estimated parameter
  \[ \sigma^2(a_i) = \int_{-\infty}^{+\infty} (a_i - \hat{a}_i)^2 \cdot p_i(a_i \mid \bar{s}) \cdot d(a_i) \]

- Confidence interval
  - Highest Posterior Density (HPD)

Minimize \((up_i - low_i)\) Subject to \(\int_{low_i}^{up_i} p_i(a_i \mid \bar{s}) \cdot d(a_i) = \beta_i\)

That maximizes the integral of \(\int_{C_\beta} p(\bar{a} \mid \bar{s})d\bar{a}\) equal to \(\beta\)
Monte Carlo Simulation

• Modular Software or Complex System

Algorithm 1: Monte Carlo approach

begin
for all $j \in [1, J]$ do // $J$ is the total number of the iteration
  for all $k \in [1, K]$ do // Generate the parameters for all the $K$ components
    $P_k \leftarrow \text{SAMPLE}\left(p_k(\Lambda_k) \mid \bar{s}\right)$
    // The function of $\text{SAMPLE}\left(p_k(\Lambda_k) \mid \bar{s}\right)$ is to draw a sample of the parameters from the posterior pdf $p_k(\Lambda_k \mid \bar{s})$, and then put the value into the vector $P_k$.
  od
  $R_s[j] \leftarrow f(P_1, P_2, \ldots, P_K)$ // function $f(P_1, \ldots, P_K)$ computes system reliability
 od
end (*Algorithm 1*)
Case: Single-Module Software

- NHPP (Nonhomogeneous Poisson Process)
  - Goel-Okumoto (GO) model (50 data)
    \[ m(t) = a[1 - \exp(-bt)] \]
  - From MEP, we get priori
    \[
p(a, b) \propto \frac{1}{2\pi\sigma_a \sigma_b} \exp\left(- \frac{(a - \mu_a)^2}{2\sigma_a^2} - \frac{(b - \mu_b)^2}{2\sigma_b^2}\right)
    \]
  - From BA,
    \[
p(a, b | \bar{s}) \propto p(a, b) \cdot p(\bar{s} | a, b)
    \]
    \[
    \propto \frac{1}{2\pi\sigma_a \sigma_b} \exp\left(- \frac{(a - \mu_a)^2}{2\sigma_a^2} - \frac{(b - \mu_b)^2}{2\sigma_b^2}\right) \cdot p(\bar{s} | a, b)
    \]
    \[
    \propto \frac{1}{2\pi\sigma_a \sigma_b} \exp\left(- \frac{(a - \mu_a)^2}{2\sigma_a^2} - \frac{(b - \mu_b)^2}{2\sigma_b^2}\right) \exp\left(- a\l[1 - \exp(-bs_{50})]\r]a^{-50} \cdot b^{-50} \exp(-b\sum_{i=1}^{50} s_i)\right)
    \]
    \[
    = 1.5032 \times 10^{81} a^{-50} b^{-50} \exp\left(- \frac{(a - 100)^2}{200} - \frac{(b - 0.001)^2}{2 \times 10^{-8}} - a[1 - \exp(-713.41b)] - 15431b\right)
    \]
Case: Single-Module Software

- Results of Uncertainty Analysis

Marginal posterior density function with respect to $a$.

Marginal posterior density function with respect to $b$. 

Marginal posterior probability density function with respect to $a'$.

Marginal posterior probability density function with respect to $b'$.
Case: Single-Module Software

• Results of Uncertainty Analysis
Case Study: Modular Software

- **Markov Model: (Two parallel Modules)**

State 1 is down and the system is up in States 2 (one module works) and 3 (two modules work).

\[
\begin{align*}
P_2'(t) &= - (\lambda + \mu)P_2(t) + 2\lambda c P_3(t), \\
P_3'(t) &= \mu P_2(t) - 2\lambda P_3(t),
\end{align*}
\]

\[
R_s(t) = P_2(t) + P_3(t)
\]
Case Study: Modular Software

• Results of Uncertainty Analysis
Case Study: Complex System

- Grid Computing
Case Study: Complex System

- Grid: Results of Uncertainty Analysis
Conclusion

• Uncertainty in Software Reliability modeling
• MEP+BA
• MC simulation
• Case Studies (Single, Modular, Complex)
Q&A